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Semigroup theory and diffusion of hadrons in the atmosphere

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Abstract. We solve the one-dimensional diffusion equation of nucleons and mesons in the atmosphere using the semigroup theory. We show that the general solutions become simplified expressions when we assume a power-law dependence on energy for the p–air inelastic cross section, $\sigma = \sigma_0 E^a$ and for the primary energy spectrum, $N_0 E^{-(\gamma+1)}$. Our solution is compared with the hadron fluxes of emulsion chamber data: we find a good consistency with an average inelasticity around 0.63, with the best fit of the coefficient $a = 0.06$.

1. Introduction

The semigroups have been successfully applied to the integration of some equations of physics such as, the Schrödinger equation, the heat conduction equation, and in other problems of quantum mechanics and quantum eletrodynamics [1]. The purpose of this paper is to show that the case of the diffusion of hadrons in the atmosphere is another example that can also be solved by this method.

These diffusion equations have been integrated with the method of the Mellin transform following the procedure of Landau and Rumer [2] to solve the integro-differential equations that describe the diffusion of the eletromagnetic cascades in the atmosphere. The solution thus obtained is represented by a contour integral which only in very few particular cases can be evaluated exactly.

We have solved these equations recently [3, 4] using the successive approximation method and we show here that the operational calculus permits us to immediately obtain the solution of the problem, saving a great amount of labour.

Assuming that the interaction mean-free path of nucleons and pions have the same power-law dependence on energy [5, 6], the pion interaction mean-free path may be obtained by multiplying the former by a constant value, $\lambda_\pi(E) = \omega_\pi \lambda(E)$ [7]. Assuming also, that the primary energy spectrum of cosmic rays is given by a JACEE energy distribution [8], we obtain the usual solution [7].

Our solution is also compared with hadron fluxes measured at mountain altitudes using an arbitary nucleon elasticity distribution [10].

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2. The nucleon diffusion equation

The diffusion of the nucleonic component in the earth's atmosphere can be described approximately by the following integro-differential equation

$$\frac{\partial N(t,E)}{\partial t} = -\frac{N(t,E)}{\lambda(E)} + \int_0^1 \frac{N(t,E/\eta)}{\lambda(E/\eta)} f(E/\eta)\frac{\mathrm{d}\eta}{\eta} \tag{1}$$

with the boundary condition

$$N(0,E)\,\mathrm{d}E = G(E)\,\mathrm{d}E \tag{2}$$

which represents the differential spectrum of the primary cosmic ray nucleons at the top of atmosphere.

In equation (1), $\lambda(E)$ and η are respectively the interaction mean-free path and the elasticity coefficient of the nucleons in the atmosphere. The elasticity coefficient is distributed according to $f(\eta)$.

About 20 years ago, Castro [9] introduced a symbolic method to solve equation (1) for the special case of η and λ constants. He introduced an operation

$$\hat{\sigma} N(t,E) = \frac{1}{\eta} N(t,E/\eta) \qquad \text{for } \eta \geqslant \eta_{\min} > 0 \tag{3}$$

where the operator $\hat{\sigma}$ works only on energy, E, and has for dominium the set of positive function $N(t,E)$, bounded and continuous with respect to E. If we suppose that $\frac{1}{\lambda(E)}$ belongs to the same dominium of $\hat{\sigma}$ and introducing the operator $\hat{A}$ independent of t,

$$\hat{A} = -\left(1 - \int_0^1 f(\eta)\,\mathrm{d}\eta\hat{\sigma}\right)\frac{1}{\lambda(E)} \tag{4}$$

in equation (1), we obtain the operator equation

$$\frac{\partial N}{\partial t}(t,E) = -\hat{A}N(t,E). \tag{5}$$

Provided that $\hat{A}$ is bounded, the solution of this equation is

$$N(t,E) = \mathrm{e}^{-t\hat{A}} N(0,E). \tag{6}$$

The operators $G_t = \mathrm{e}^{-t\hat{A}}$, for $t \geqslant 0$, are the elements of a semigroup $\{G_t\}$ [1], with

$$\begin{aligned} &G_t \cdot G_r = G_{(t+r)} \\ &G_{t=0} = \mathbb{1} \qquad t, r \geqslant 0 \end{aligned} \tag{7}$$

where G_0 is the identity operator and $\hat{A}$ are the generators of this semigroup.

The operator $\hat{A}$ is the sum of two operators,

$$A_1 = -\frac{1}{\lambda(E)} \qquad \text{and} \qquad A_2 = \left(\int_0^1 f(\eta)\,\mathrm{d}\eta\hat{\sigma}\right)\frac{1}{\lambda(E)}.$$

The operators $\hat{A}_1$ and $\hat{A}_2$, in general, do not commute and only in the particular case, $\lambda(E) = \lambda_0 =$ constant we will have $\exp(\hat{A}_1 + \hat{A}_2) = \exp(\hat{A}_1)\exp(\hat{A}_2)$.

For the general case we must consider the order of the factors in the development of the $\exp(\hat{A}_1 + \hat{A}_2)$, in the power series

$$G_t = \sum_{n=0}^{\infty} (-1)^n (\hat{A}_1 + \hat{A}_2)^n \frac{t^n}{n!} \tag{8}$$

with $\hat{A}_1\hat{A}_2 \neq \hat{A}_2\hat{A}_1$.

Bellandi *et al* [10] recently used a similar formal operator to solve equation (1).

3. The meson diffusion equation

The diffusion of mesons ‘m’ in the atmosphere can be written

$$\frac{\partial M(t,E)}{\partial t} = -\frac{M(t,E)}{\lambda_m(E)} + \int_0^1 \frac{M(t,E/x)}{\lambda_m(E/x)} f_{mm}(x)\frac{\mathrm{d}x}{x} + \int_0^1 \frac{N(t,E/x)}{\lambda(E/x)} f_{nm}(x)\frac{\mathrm{d}x}{x} \tag{9}$$

with the boundary condition

$$M(0,E) = 0 \tag{10}$$

where $\lambda_m(E)$ is the interaction mean-free path of the meson in the atmosphere; f_{mm} and f_{nm} are respectively the spectra of the mesons produced in the meson–air nuclei and in the nucleon–air nuclei interactions, and x is the Feynman variable which for high energy is approximately $x \cong \frac{E}{E'}$ (E' is the primary energy of nucleon or meson).

As in the nucleon case, in order to solve the diffusion equation (9) for the mesons, we introduce the operators

$$\hat{B}_N = \left(\int_0^1 f_{nm}(x)\,\mathrm{d}x\,\hat{\sigma}_n\right)\frac{1}{\lambda(E)} \tag{11}$$

and

$$\hat{B}_m = -\left(1 - \int_0^1 f_{mm}(x)\,\mathrm{d}x\,\hat{\sigma}_m\right)\frac{1}{\lambda_m(E)}. \tag{12}$$

B_N and B_m as defined above operate only on the energy, E, and have for dominium the set of positive functions $N(t,E)$ and $M(t,E)$ bounded and continuous with respect to E, in the range, $0 < E_{\min} < \infty$. If we suppose that $\frac{1}{\lambda_m(E)}$ belongs to the same dominium of $\hat{\sigma}_m$, the equation (9) takes the following form:

$$\frac{\partial M(t,E)}{\partial t} = \hat{B}_m M(t,E) + \hat{B}_n N(t,E). \tag{13}$$

The formal solution of operator equation (12) that satisfies the boundary condition (10) is

$$M(t,E) = \int_0^t \mathrm{e}^{-(t-z)\hat{B}_m}\hat{B}_n N(z,E)\,\mathrm{d}z. \tag{14}$$

Similarly the operators $\hat{H}_t = \mathrm{e}^{-t\hat{B}_m}$ for $t > 0$, are the elements of a semigroup $\{H_t\}$. The generators $\hat{B}_m$ of this semigroup are the sum of two operators $\hat{B}_{m_1} = -\frac{1}{\lambda_m(E)}$ and $\hat{B}_{m_2} = (\int_0^1 f_m(x)\,\mathrm{d}x\,\hat{\sigma}_m)\frac{1}{\lambda_m(E)}$. These operators as in the nucleon case, in general, do not commute.

4. Particular case

Taking the cosmic-ray primary energy spectrum $N(0,E) = N_0 E^{-(\gamma+1)}$ [8], and if the interaction mean-free path for nucleons decreases with energy in the form $\lambda(E) = \lambda_0 E^{-a}$ [5], the solutions (6) and (14) will take the simplified expressions as follows.

4.1. Differential nucleon flux

The differential nucleon fluxes can be written

$$N(t,E) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{t}{\lambda(E)}\right)^n I_n(\gamma,a,n)N(0,E) \tag{15}$$

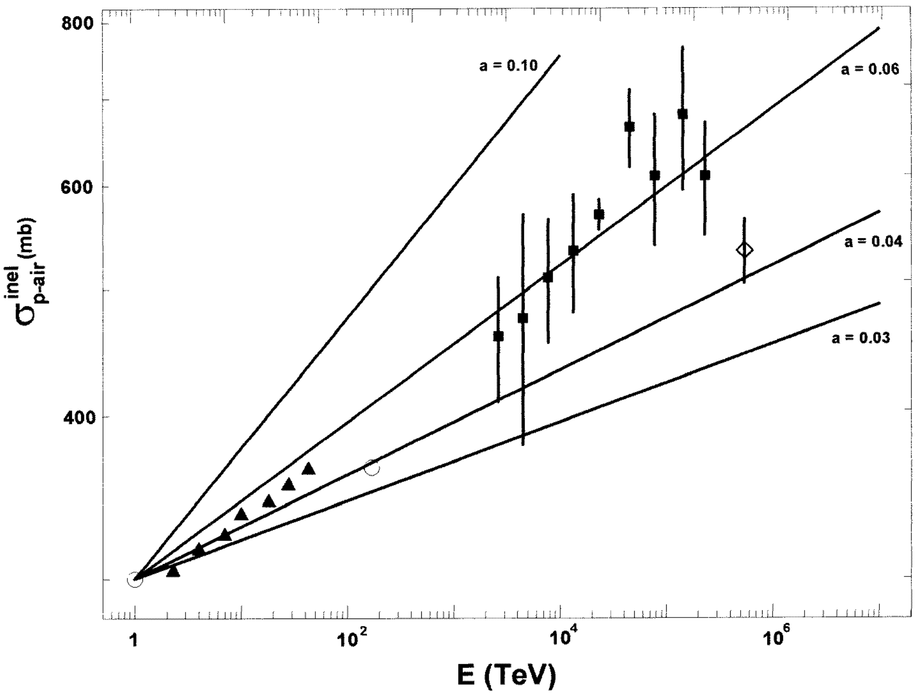

Figure 1. Inelastic cross sections of p–air against energy. Data are from [11] △, □ and ◊, [12] for ○. Four full lines are drawn for $a = 0.03$, 0.04, 0.06 and 0.10 in the formula $\sigma = 300(E/\mathrm{TeV})^a$.

with

$$I_n(\gamma, a, n) = \prod_{j=1}^{n}(1 - \langle\eta^{\gamma-ja}\rangle) \tag{16}$$

and

$$\langle\eta^{\gamma-aj}\rangle = \int_0^1 f(\eta)\,\mathrm{d}\eta\,\eta^{\gamma-aj}. \tag{17}$$

If $a = 0$ we obtain the well known solution for λ independent of E.

The expression (15) is the same solution that appears in [10].

4.2. *Differential meson flux*

The meson flux (14) takes the following form

$$M(t, E) = \sum_{k=0}^{\infty}\int_0^t (-1)^k \frac{(t-z)^k}{k!}\left(\hat{B}_m \frac{1}{\lambda_m(E)}\right)^k \cdot B_n N(t, E) \tag{18}$$

where $\hat{B}_n N(t, E)$ is the production rate of secondary mesons 'm' by the nucleon–air nuclei interactions and it is assumed the expression

$$\hat{B}_n N(t, E) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{z^n}{\lambda(E)^{n+1}} I_n(\gamma, a, n) Z_{nm}(\gamma, a, n) N_0 E^{(\gamma+1)} \tag{19}$$

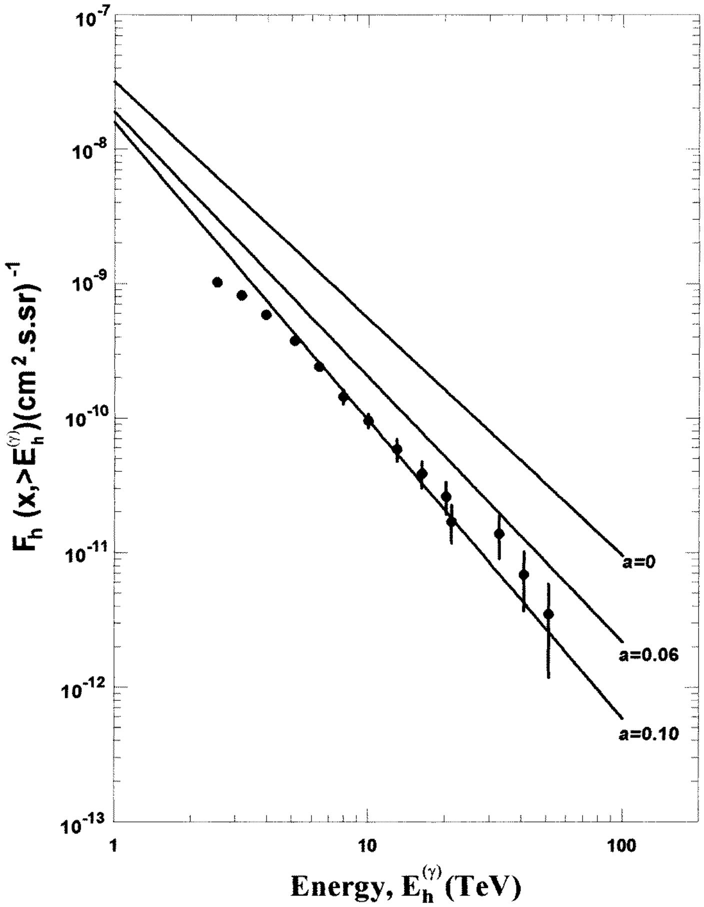

Figure 2. Integral hadron spectrum at 540 g cm^{-2}. (● from [17]). The three lines are drawn for $\langle K\rangle = 0.5$ with $a = 0$, 0.06 and 0.10.

where $Z_{nm}(\gamma, a, n)$ is the energy-weighted spectrum for nucleons interacting with air nuclei for $\lambda(E) = \lambda_0 E^{-a}$, and assume the expression

$$Z_{nm}(\gamma, a, n) = \int_0^1 x^{\gamma-(n+1)a} f_{nm}(x)\, \mathrm{d}x. \tag{20}$$

If $\lambda(E)$ and $\lambda_m(E)$ have the same power-law dependence on energy [7], and so $\lambda_m(E) = \omega_m(E)$ with ω_m = constant, then the expression (18) takes the form

$$M(t, E) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\int_0^t \mathrm{d}z\, \frac{(-1)^k(-1)^n}{k!n!}\left(\frac{(t-z)}{\lambda_m(E)}\right)^k\left(\frac{z}{\lambda(E)}\right)^n \times Z_{nm}(\gamma, a, n) I_n(\gamma, a, n) I_{mm}(\gamma, a, k, n)\frac{N_0 E^{-(\gamma+1)}}{\lambda(E)} \tag{21}$$

where

$$I_{mm}(\gamma, a, k, n) = \prod_{i=1}^{k}(1 - \langle x^{\gamma-a(k+n+1)}\rangle) \tag{22}$$

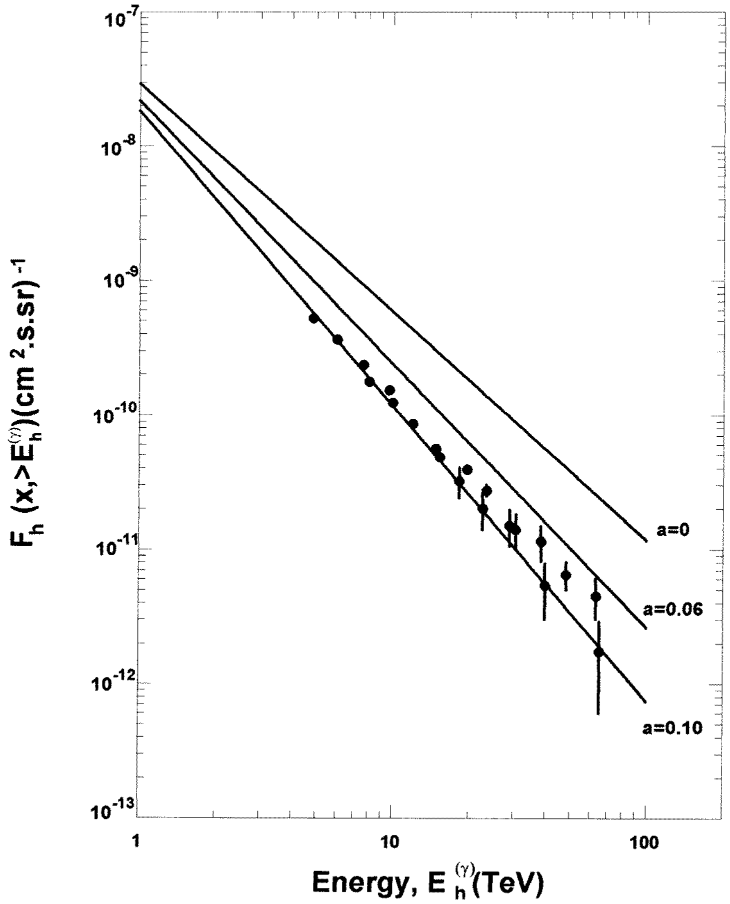

Figure 3. Integral hadron spectrum at 520 g cm^{-2}. (● from [18]). The three lines are drawn for $\langle K\rangle = 0.5$ with $a = 0$, 0.06 and 0.10.

with

$$\langle x^{\gamma-a(k+n+1)}\rangle = \int_0^1 f_{mm}(x)x^{\gamma-a(k+n+1)}\,\mathrm{d}x. \tag{23}$$

In the special case when $a = 0$ the expressions (16), (22) and (20) become

$$I_n(\gamma, a, n) = (1 - \langle\eta^\gamma\rangle)^n \tag{24}$$

$$I_{mm}(\gamma, a, k, n) = (1 - \langle x^\gamma\rangle)^n \tag{25}$$

with

$$\langle x^\gamma\rangle = \int_0^1 f_{mm}(x)x\gamma\,\mathrm{d}x \tag{26}$$

and

$$Z_{nm} = \int_0^1 f_{nm}(x)x^\gamma\,\mathrm{d}x. \tag{27}$$

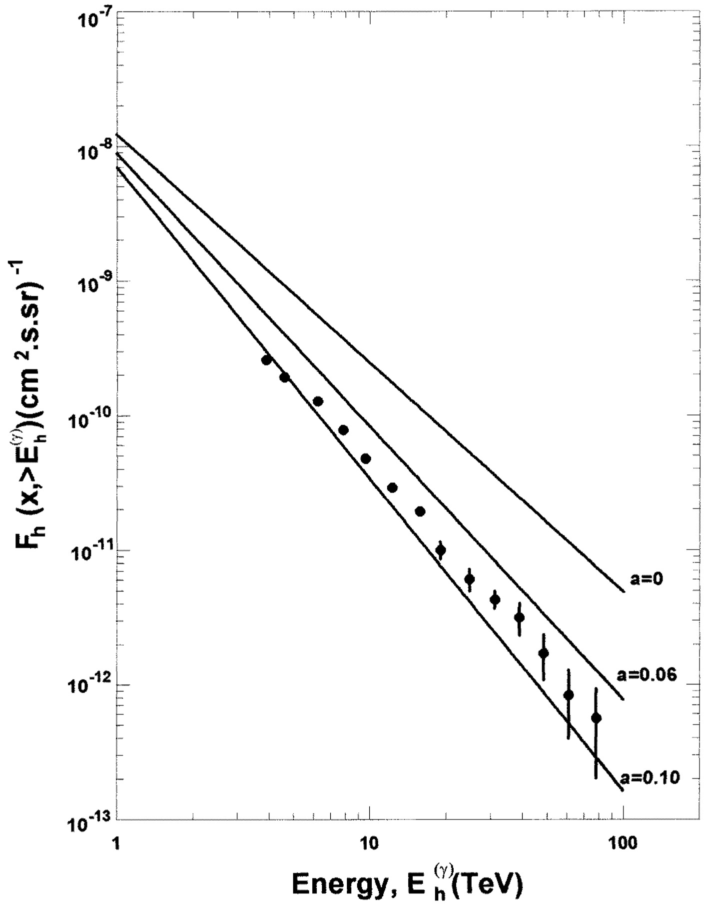

Figure 4. Integral hadron spectrum at 650 g cm^{-2}. (● from [19]). The three lines are drawn for $\langle K\rangle = 0.5$ with $a = 0$, 0.06 and 0.10.

So solution (21) reduces to the expression corresponding to λ independent of E.

$$M(t, E) = N_0 E^{-(\gamma+1)} \frac{Z_{nm}}{\lambda} \frac{e^{-t/L_m} - e^{-t/L}}{1/L - 1/L_m} \tag{28}$$

where L and L_m are respectively the absorption mean-free path of nucleons and mesons 'm' in the atmosphere, with $L = \frac{\lambda}{1-\langle\eta^\gamma\rangle}$ and $L_m = \frac{\lambda_m}{1-\langle x^\gamma\rangle}$.

5. Comparison with experimental data

In order to make a comparison with hadron fluxes measured at mountain altitudes with emulsion chambers, we need to take into account various elements in addition to the cross section, like the primary cosmic-ray flux, the distribution of elasticity, the γ-ray inelasticity and the energy spectra in the laboratory system for the secondary mesons.

We shall make the following simple considerations on each element.

(a) Primary cosmic rays: at the atmospheric top, the majority of incoming cosmic ray particles are protons. The number of nuclei cannot, however, be neglegible in order

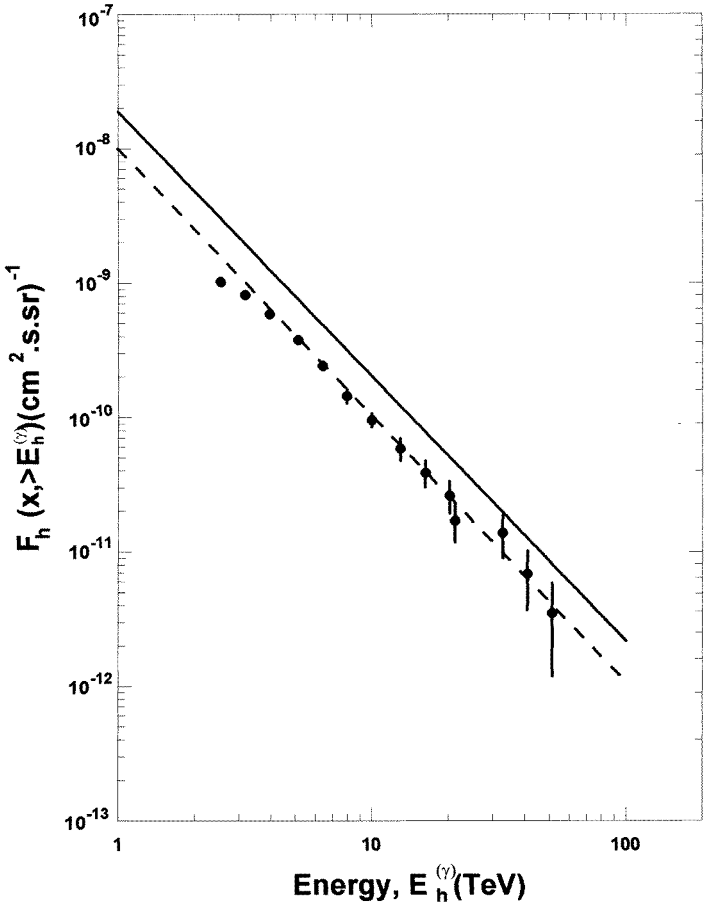

Figure 5. Integral hadron spectrum at 540 g cm^{-2}. (● from [17]). The full line represents the calculated flux for $\langle K \rangle = 0.5$ and the broken line is the same flux for $\langle K \rangle = 0.63$. Both lines are for $a = 0.06$.

to study the hadron flux. Bhattacharyya [8] analysed experimental data of balloon-borne experiments and reported the nucleon flux at the top of the atmosphere to be $2.237E^{-2.7}$ (cm^2 s sr GeV/nucleon)$^{-1}$.

(b) Nucleon elasticity distribution: we assume for the nucleon elasticity distribution the following arbitrary form [10], $f(\eta) = (1+\beta)(1-\eta)^{\beta}$ in the interval 0–1. This distribution did not take into account the diffractive phenomena. The flat distribution corresponds to the case $\beta = 0$.

(c) γ-ray inelasticity: in emulsion chamber experiments, hadrons are detected as cascade showers. Thus the measured energy $E_h^{(\gamma)}$ is related to the hadron energy E_h as $E_h^{(\gamma)} = K_\gamma E_h$ where K_γ is the γ-ray inelasticty. We use $\langle K_\gamma \rangle = 0.25$ as the effective inelasticity in the experiments [15].

(d) Secondary pions: we assume that only the pions are generated in the multiparticle production, neglecting the particles of small fractions such as kaons, heavy mesons, etc. The energy spectra of mesons in the laboratory system, $f_{nm}(x)$ and $f_{mm}(x)$ are obtained from the accelerator data [16] assuming a scaling-type pion production. The expression of

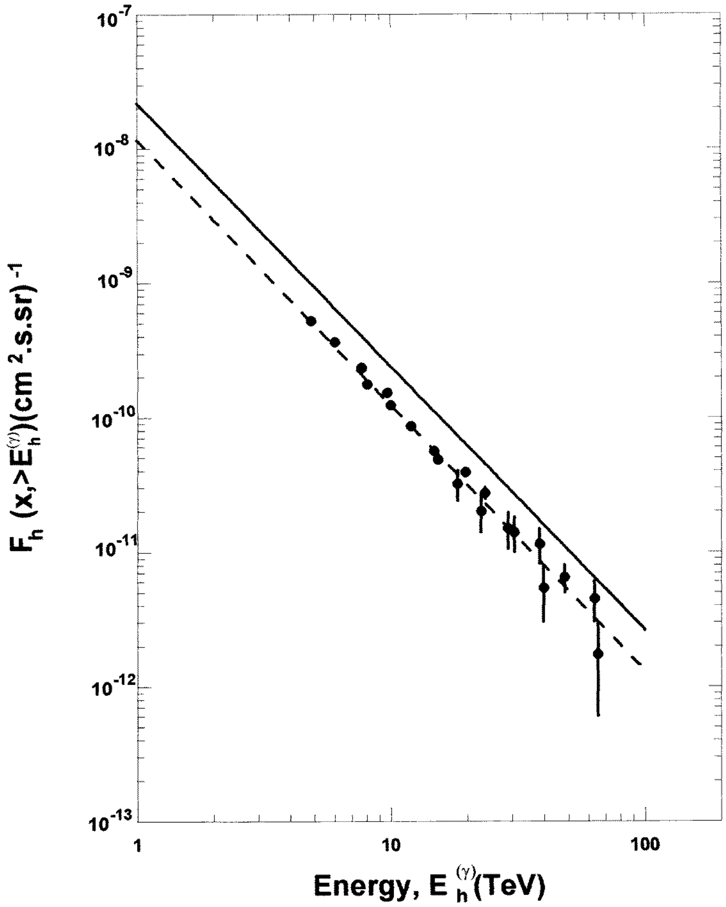

Figure 6. Integral hadron spectrum at 520 g cm^{-2}. (● from [18]). The full line represents the calculated flux for $\langle K \rangle = 0.5$ and the broken line is the same flux for $\langle K \rangle = 0.63$. Both lines are for $a = 0.06$.

these spectra are

$$f_{n\pi}(x) = 1.04 \frac{1-x}{x} \mathrm{e}^{-5x} \qquad \text{for } N + \text{air} \longrightarrow \pi + \text{anything} \tag{29}$$

and

$$f_{\pi\pi}(x) = \frac{1.3}{x}\left(1 + \frac{x}{0.45}\right)^{-3} + \frac{0.16}{x}\mathrm{e}^{2(x-1)} \text{for only one charge-pion state.} \tag{30}$$

(e) p–air inelastic cross section: several functional forms have been proposed to fit the behaviour of rising cross section, among which we adopt the following one in our calculation

$$\sigma = \sigma_0 (E/\mathrm{TeV})^a. \tag{31}$$

In figure 1, the cross sections of inelastic interactions between protons and air nuclei are plotted against energy. Data are from air-shower experiments [11] and from accelerator experiments [12]. For the last ones, σ(p–p) and $\sigma(\bar{\mathrm{p}}$–p) are converted into σ(p–air) by the empirical formula of Hillas [13]. Four cases of the power-law energy-dependent cross

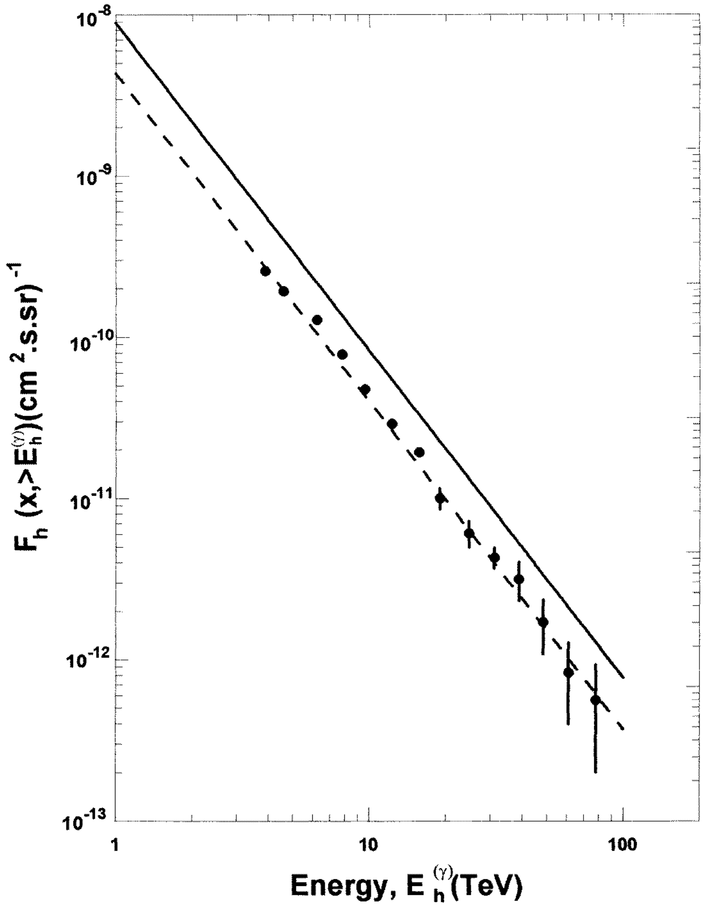

Figure 7. Integral hadron spectrum at 650 g cm^{-2}. (● from [19]). The full line represents the calculated flux for $\langle K\rangle = 0.5$ and the broken line is the same flux for $\langle K\rangle = 0.63$. Both lines are for $a = 0.06$.

section are shown by full lines for a guide, for a = 0.03, 0.04, 0.06 and 0.10. The best values are $a = 0.06$ and $\sigma_0 = 300$ mbarn.

If we use another convertion, for example, that of Kopeliovich *et al* [14], we continue to have $a = 0.06$ but σ_0 becomes 293 mbarn.

Figures 2–4 show the comparison of our solution with the integral hadron fluxes measured at Chacaltaya (540 g cm^{-2}), Kanbala (520 g cm^{-2}) and Fuji (650 g cm^{-2}), respectively. Three curves of $a = 0$, 0.06 and 0.10 are also drawn in the figures for each depth taking into account the above-mentioned items (a)–(e) and using a flat distribution for nucleon elasticity. We see in the figures that the experimental data are between the curves $a = 0.06$ and 0.10.

Figures 5–7 show the same type of comparison, with the curves of $a = 0.06$ and taking into account the usual mean value of inelasticity coefficient $\langle K\rangle = 0.5$ and an arbitrary nucleon elasticity distribution, with $\langle K\rangle = 0.63$. We see that the best agreement in the three figures is $\langle K\rangle = 0.63$.

6. Discussions and conclusions

We have solved the diffusion equations of cosmic ray nucleons and mesons analytically using the semigroup theory and taking into account the rising of the cross section with the energy in a general way. The solutions are written in the compact expression (6) and (14). These solutions become the simplified forms when we assume a power-law dependence on energy for the hadron-air cross sections and for the primary energy spectrum. The hadron fluxes at mountain atmospheric depths decrease when we include in our calculation the rising of the cross section and the decreasing of the average nucleon elasticity.

Through a comparison with the integral hadron fluxes at mountain altitudes, we have found that $\langle K \rangle = 0.63$ gives a good consistency when we use the best value of the coefficient a (0.06). Our result is in agreement with that of Jones [20] based in an analysis on inclusive reactions data from accelerator. There is also an agreement with analytical calculations about nucleon fluxes at sea level [10, 21].

The hadron fluxes are largely affected when the coefficient a changes in the interval 0.03–0.10. In this case the best fit of our calculation with experimental hadron fluxes is obtained for $\langle K \rangle$ changing between 0.72 and 0.45.

The effect on the integral hadron fluxes is very small when we use other well known values for the nucleon and pion mean-free paths. For example, if we use $\lambda = 83$ g cm^{-2} ($\sigma_0 = 293$ mbarn) and $\lambda_\pi = 116$ g cm^{-2} ($\sigma_\pi = 1.4\sigma_0$) at 1 TeV the best fit, gives $\langle K \rangle = 0.61$ for $a = 0.06$.

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